Random-matrix universality in the small-eigenvalue spectrum of the lattice Dirac operator

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We analyze complete spectra of the lattice Dirac operator in SU(2) gauge theory and demonstrate that the distribution of low-lying eigenvalues is described by random matrix theory. We present possible practical applications of this random-matrix universality. In particular, reliable extrapolations of lattice gauge data to the thermodynamic limit are discussed.

1. Introduction

Based on an analysis of sum rules derived by Leutwyler and Smilga [1] for inverse powers of the eigenvalues of the QCD Dirac operator in a finite volume, Shuryak and Verbaarschot [2] conjectured that the so-called microscopic spectral density of the Dirac operator, $\rho_s(z)$, is a universal function which can be computed in random matrix theory (RMT). This quantity is defined by

$$\rho_s(z) = \lim_{V\to\infty} \frac{1}{V\Sigma}\rho(\frac{z}{V\Sigma})\,, \qquad (1)$$

where $\rho(\lambda) = \langle \sum_n \delta(\lambda - \lambda_n)\rangle_A$ is the spectral density of the Dirac operator averaged over all gauge field configurations A, V is the space-time volume, and Σ is the absolute value of the chiral condensate. The chiral condensate can be determined using the Banks-Casher relation [3]

$$\langle\bar\psi\psi\rangle = \lim_{m\to 0}\lim_{V\to\infty} \frac{\pi\rho(0}{V}\,. \qquad (2)$$

The definition of Eq. (1) amounts to a magnification of the region of low-lying eigenvalues by a factor of the volume. Since the spacing of small eigenvalues is $\sim 1/(V\Sigma)$, this definition leads to the resolution of individual eigenvalues.

*Talk presented by T. Wettig

A number of studies have presented evidence supporting the universality conjecture for ρ_s. They are summarized in Ref. [4]. Here, we present and analyze the results of a high-statistics lattice study in SU(2) gauge theory with staggered fermions which confirms this conjecture in a particularly direct way. Most of the details omitted here, in particular regarding the lattice calculations, can be found in Ref. [4], in the talk by S. Meyer, and in the poster by M.E. Berbenni-Bitsch, respectively (these proceedings).

2. Microscopic universality

Depending on the number of colors and on the representation of the fermions, one must distinguish three different universality classes which were classified in Ref. [5]. These correspond to the three chiral ensembles of RMT, the chiral Gaussian orthogonal (chGOE), unitary (chGUE), and symplectic (chGSE) ensemble. Analytic random-matrix results have been obtained for all three ensembles for a variety of quantities including the microscopic spectral density, the distribution of the smallest eigenvalue, and microscopic

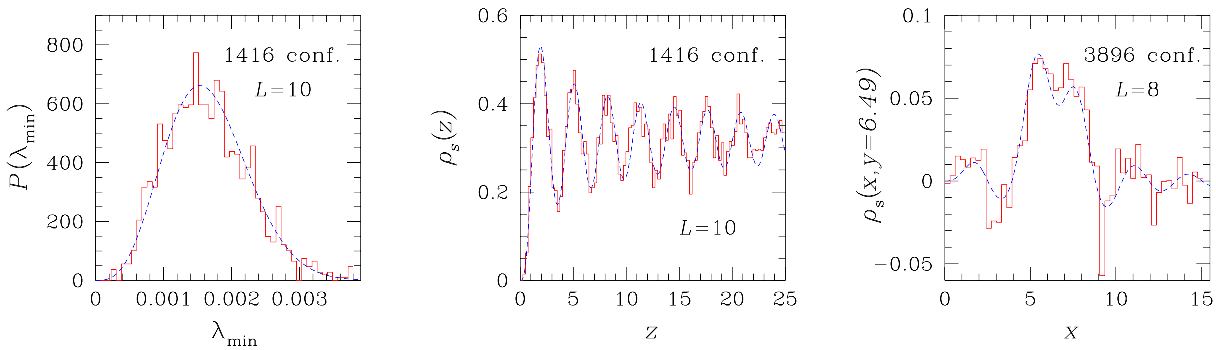

Figure 1. The distribution of the smallest eigenvalue $P(\lambda_{\rm min})$ and microscopic spectral density $\rho_s(z)$ (for a 10^4 lattice) and the microscopic spectral two-point function $\rho_s(x,y)$ (for an 8^4 lattice.) The histograms represent lattice data; the dashed lines are analytical predictions from random matrix theory.

spectral correlators. We have worked with staggered fermions in SU(2) for which the appropriate RMT ensemble is the chGSE. RMT results for the chGSE can be obtained by a slight modification of the analytic results obtained by Forrester [6] and by Nagao and Forrester [7] for generalized Laguerre ensembles. In particular, we have

$$\rho_s(z) = z[J_0^2(2z)+J_1^2(2z)] - \frac{1}{2}J_0(2z)\int_0^{2z} dt J_0(t) \tag{3}$$

for the microscopic spectral density and

$$P(\lambda_{\rm min}) = \sqrt{\frac{\pi}{2}}c(c\lambda_{\rm min})^{3/2}I_{3/2}(c\lambda_{\rm min})e^{-\frac{1}{2}(c\lambda_{\rm min})^2} \tag{4}$$

for the distribution of the smallest eigenvalue. Here, J (and I) are (modified) Bessel functions, and $c = V\langle\bar\psi\psi\rangle$. Note that Eq. (3) is simpler than the equivalent expression quoted in Ref. [4]. The RMT result for the connected microscopic spectral two-point correlator is

$$\rho_s(x,y) = f\partial_x\partial_y f - \partial_x f\partial_y f\ , \tag{5}$$

where

$$f(x,y) = \frac{y}{2}\int_0^{2x} dt\, C(t,2y) - \frac{x}{2}\int_0^{2y} dt\, C(t,2x)$$

with

$$C(x,y) = \frac{xJ_1(x)J_0(y)-yJ_0(x)J_1(y)}{x^2-y^2}\ .$$

We have performed lattice simulations using $\beta = 2.0$ for four different lattice sizes L^4 with $L = 4,6,8,10$. Lattice data for all three quantities are compared with the RMT predictions of Eqs. (3)–(5) in Fig. 1 for selected lattice sizes. The number of configurations is indicated in the figure. The agreement is remarkably good even for such modest lattice sizes. Note that there are no free parameters. The only parameter which enters the RMT predictions is given by the volume and the chiral condensate, which has been fixed by the lattice data using the Banks-Casher relation.

Since the RMT results are obtained in the thermodynamic limit, the quality of the agreement between lattice data and RMT increases with increasing physical volume. Hence, for larger values of β, one requires larger lattices to attain the same level of agreement. We have confirmed this expectation using $\beta = 2.2$ on 6^4 and 8^4 lattices and $\beta = 2.5$ on a 16^4 lattice, respectively.

3. Practical applications

Having convinced ourselves and, hopefully, the reader that the distribution of low-lying eigenvalues in the limit of large V is given exactly by RMT, we now ask how one can make practical use of this knowledge. One immediate application is the following. We have pointed out earlier that the RMT predictions are parameter-free; the only parameter appearing in the expressions (essen-

Table 1
The chiral condensate as a function of the lattice size, determined by the Banks-Casher relation (BC) and by a fit to the random-matrix prediction (RMT). See also Fig. 2.

L	$\langle\bar\psi\psi\rangle$ (BC)	$\langle\bar\psi\psi\rangle$ (RMT)	χ^2/dof
4	0.1131(19)	0.1262(17)	6.75
6	0.1209(16)	0.1263(12)	1.87
8	0.1228(25)	0.1255(12)	1.78
10	0.1247(22)	0.1256(10)	1.15

tially the chiral condensate) is fixed by the lattice data. Given the agreement between lattice data and RMT, one can turn the argument around and determine the chiral condensate by adjusting the energy scale in order to fit the lattice data for, say, the microscopic spectral density to the RMT prediction of Eq. (3). This procedure yields an independent estimate of the chiral condensate which should be compared to the value obtained from the Banks-Casher relation. This comparison is made in Table 1 where we also include the χ^2 per degree of freedom for the fit to Eq. (3). The same numbers are displayed in Fig. 2. The results are striking. For all lattice sizes considered, the values of $\langle\bar\psi\psi\rangle$ obtained from the fit to RMT agree with each other and with the thermodynamic limit within error bars. The values obtained from the Banks-Casher relation approach the thermodynamic limit more slowly. The surprising result of this analysis is that with a better accounting of finite size effects with random matrix theory, the chiral condensate is remarkably volume independent. This approach appears to offer an interesting technical advance by making it possible to extract certain information about the thermodynamic limit from small lattices. One can also extract scalar susceptibilities from a similar analysis of spectral two-point functions; work in this direction is in progress. Precise numbers for the condensate and the susceptibilities as a function of temperature are needed to determine critical exponents close to the chiral phase transition. We are currently testing how the proposed approach works at finite temperatures, in particular close to the critical temperature.

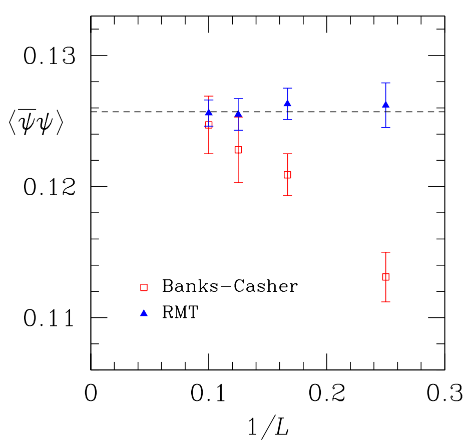

Figure 2. Extrapolation of $\langle\bar\psi\psi\rangle$ to the thermodynamic limit (indicated by the dashed line), via the Banks-Casher relation (boxes) and by fitting the lattice data for $\rho_s(z)$ to Eq. (3) (triangles).

4. Summary

We believe to have demonstrated unambiguously that the distribution of low-lying eigenvalues of the Dirac operator is described by RMT. Some practical applications of this fact have been discussed in Sec. 3. We are confident that more can be found.

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